Remembering Aldo de Luca

Alberto Pettorossi
University of Roma "Tor Vergata", Roma, Italy
CNR-IASI, Roma, Italy

University "Sapienza", Roma, Italy

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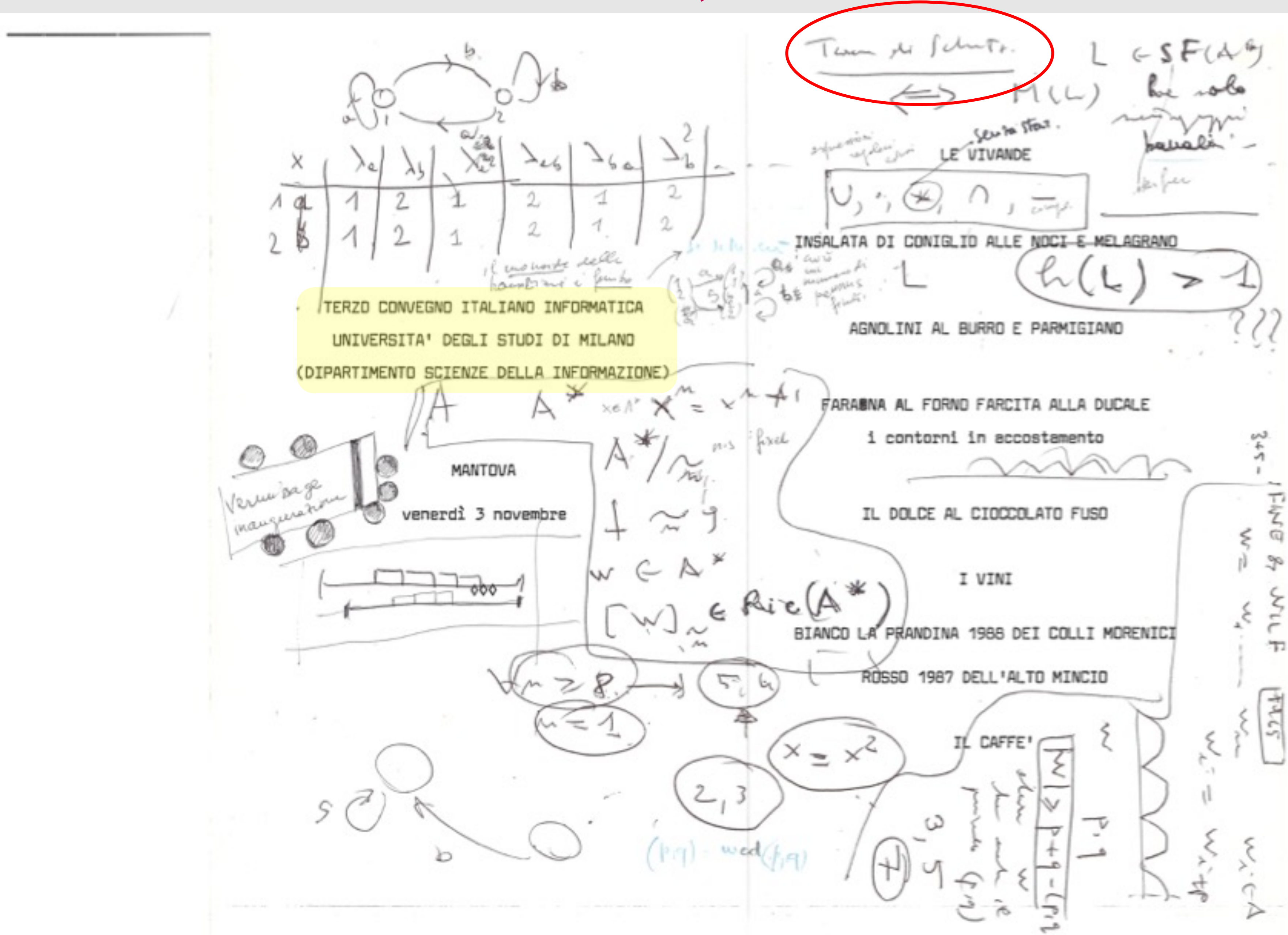


TERZO CONVEGNO ITALIANO INFORMATICA

UNIVERSITA' DEGLI STUDI DI MILANO

(DIPARTIMENTO SCIENZE DELLA INFORMAZIONE)

MANTOVA

venerdì 3 novembre

LE VIVANDE

INSALATA DI CONIGLIO ALLE NOCI E MELAGRANO

AGNOLINI AL BURRO E PARMIGIANO

FARAONA AL FORNO FARCITA ALLA DUCALE

i contorni in accostamento

IL DOLCE AL CIOCCOLATO FUSO

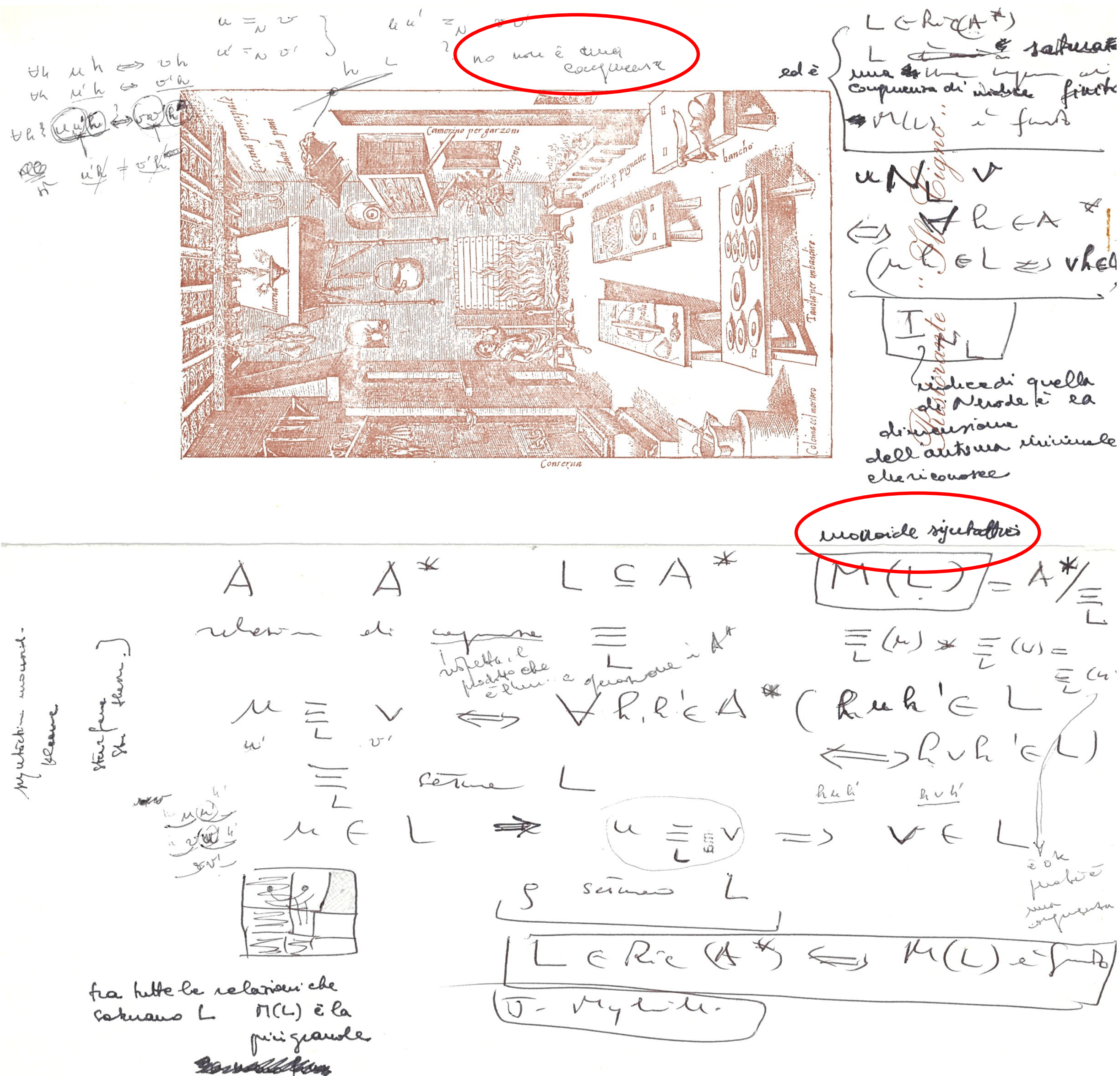
I VINI

BIANCO LA PRANDINA 1988 DEI COLLI MORENICI

ROSSO 1987 DELL'ALTO MINCIO

IL CAFFE'

Aldo: Schützenberger Theorem and Green Relations



Aldo: Fibonacci words

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A COMBINATORIAL PROPERTY OF THE FIBONACCI WORDS

Aldo de LUCA
Istituto di Matematica dell'Università di Napoli, Italy and Istituto di Cibernetica del CNR, Arco Felice, Napoli, Italy

Received 28 March 1981

Fibonacci words, palindrome words

1. Introduction

In the combinatorial theory of free monoids the sequence of words of Fibonacci plays a very important role since the words of Fibonacci have remarkable combinatorial properties some of which have been stressed by Knuth [4] in relation with problems of 'string matching' and, more recently, by Duval [3] in the study of 'periodicity' of the words.

In this paper by making use of a result of Berstel (cf. Proposition 1) which states that for $n \geqslant 3$ the Fibonacci words f_n have a palindrome left-factor of length $|f_n| - 2$, we shall prove that (cf. Proposition 2) for all $n > 4$, f_n *is the product of two, uniquely determined, palindrome words of lengths* $F(n-1) - 2$ *and* $F(n-2) + 2$, where $F(n) = |f_n|$ is the n-th term of the Fibonacci numerical sequence.

These two properties of the Fibonacci words are of great interest since we can show (cf. Proposition 3) that for $n > 4$, *the Fibonacci sequence* f_n *is the unique sequence of words satisfying the previous properties and the additional requirements that the words contain at least two different letters and that they always begin with a same letter* (the letter 'b' in our case).

any $w = a_1 \cdots a_n$, $a_i \in A$, $1 \leqslant i \leqslant n$, the *reversed word* $\tilde{w}$ is defined as $\tilde{w} = a_n \cdots a_1$. Moreover $\tilde{1} = 1$. A word w is called *palindrome* if $w = \tilde{w}$.

In the following we consider an alphabet A whose cardinality $|A|$ is $\geqslant 2$. The sequence $\{f_n\}$, $n \geqslant 1$, of words of Fibonacci is defined inductively as:

$$f_1 = a, \quad f_2 = b, \quad f_{n+1} = f_n f_{n-1},$$

$$a, b \in A, \quad a \neq b, \quad n \geqslant 2.$$

The length $|f_n|$ of f_n is the n-th term F(n) of the numerical sequence of Fibonacci since $|f_1| = |f_2| = 1$ and $|f_{n+1}| = |f_n| + |f_{n-1}|$ for all $n \geqslant 2$.

Proposition 1. For all $n \geqslant 3$ one has that $f_n = \alpha_n d_n$, where α_n is palindrome and $d_n = ab$ if n is even and $d_n = ba$ if n is odd.

Proof. The proof is by induction on the integer n. The result is trivial for n = 3 and n = 4. Let us then suppose that $n > 4$. One has that:

$$f_n = f_{n-1} f_{n-2} = f_{n-2} f_{n-3} f_{n-2}$$

$$= \alpha_{n-2} d_{n-2} \alpha_{n-3} d_{n-3} \alpha_{n-2} d_{n-2}.$$

Fibonacci word: $w_{n+2} = w_{n+1} + w_n$

1 a
2 b
3 = 2 1 b a
4 = 3 2 ba b
5 = 4 3 bab ba
6 = 5 4 babba bab
7 = 6 5 babbabab babba
8 = ...

Aldo:
palindrome ab or palindrome ba

1 a
2 b
3 = 2 1 ba
4 = 3 2 bab
5 = 4 3 babba
6 = 5 4 babbabab
7 = 6 5 babbababbabba
8 = ...

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2. The Fibonacci words – A combinatorial property

Let A be a finite, nonempty set, or *alphabet* and A^* the *free monoid* generated by A. The elements of A are called *letters* and those of A^* *words*. The identity element of A^* is denoted by 1. Further $A^+ = A^* \backslash \{1\}$ is the *free semigroup* generated by A.

For any word $w \in A^*$, $|w|$ denotes its *length*. For any $w = a_1 \cdots a_n$, $a_i \in A$, $1 \leqslant i \leqslant n$, the *reversed word* $\tilde{w}$ is defined as $\tilde{w} = a_n \cdots a_1$. Moreover $\tilde{1} = 1$. A word w is called *palindrome* if $w = \tilde{w}$.

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Proof. The proof is by induction on the integer n. The result is trivial for $n = 3$ and $n = 4$. Let us then suppose that $n > 4$. One has that:

$$f_n = f_{n-1} f_{n-2} = f_{n-2} f_{n-3} f_{n-2}$$
$$= \alpha_{n-2} d_{n-2} \alpha_{n-3} d_{n-3} \alpha_{n-2} d_{n-2}.$$

Since the words α_{n-2} and α_{n-3} are palindrome by the hypothesis of the induction and, moreover, for all n, $d_n = \tilde{d}_{n+1}$ it follows that the word

$$\alpha_n = \alpha_{n-2} d_{n-2} \alpha_{n-3} d_{n-3} \alpha_{n-2},$$

is palindrome. Thus being $d_n = d_{n-2}$ the result follows.

Proposition 2. For all $n > 4$, f_n is the product $u_n v_n$ of two uniquely determined palindrome words of A^+ whose lengths are $|u_n| = F(n-1) - 2$ and $|v_n| = F(n-2) + 2$.

Proof. By Proposition 1 one can write for all $n \geq 5$

$f_n = f_{n-1} f_{n-2} = \alpha_{n-1} d_{n-1} \alpha_{n-2} d_{n-2}$.

Since $d_{n-1} = \tilde{d}_{n-2}$ one has that f_n is the product of the two palindrome words $u_n = \alpha_{n-1}$ and $v_n = d_{n-1}\alpha_{n-2}d_{n-2}$ whose lengths are respectively $|u_n| = |\alpha_{n-1}| = |f_{n-1}| - 2 = F(n-1) - 2$, $|v_n| = 2 + |\alpha_{n-2}d_{n-2}| = 2 + |f_{n-2}| = 2 + F(n-2)$.

We shall now prove that the Fibonacci words f_n are *primitive* (i.e. for each $n \geq 1$, $f_n \neq w^p$ with $w \in A^+$ and $p \geq 2$). This will imply that the previous factorization $f_n = u_n v_n$ in two palindrome words is unique. Obviously f_1 and f_2 are primitive. Let us then suppose $n \geq 3$ and $f_n = w^p$ with $p \geq 2$ and $w \in A^+$. By Proposition 1 we can write $f_n = w^p = \alpha_n d_n$. From the definition of Fibonacci words it follows that $|w| > 1$ so that $w = w_1 w_2$ with $w_2 = d_n$ and

$\alpha_n = w^{p-1} w_1 = (w_1 w_2)^{p-1} w_1 = (\tilde{w}_1 \tilde{w}_2)^{p-1} \tilde{w}_1$.

This implies that $\tilde{w}_2 = w_2$, i.e. $\tilde{d}_n = d_n$ which is absurd. Thus Fibonacci words are primitive.

The result follows by the fact that if a primitive word is the product of two palindrome words of A^+ then this factorization is unique (cf. [2]).

The next proposition shows that the properties of the Fibonacci words expressed by Proposition 1 and 2 and the fact that these words for $n \geq 3$, contain two different letters and that for $n > 1$ the first letter is always 'b' characterize them completely. More precisely it holds the following:

Proposition 3. Let $\{w_n\}$, $n \geq 1$, be a sequence of words of A^+ each of which contains at least two different letters of the alphabet A (i.e. $\text{alph}(w_n) \geq 2$). Let us moreover suppose that for all $n \geq 5$

$w_n = \alpha_n \beta_n = \gamma_n c_n$,

with $c_n \in A^*$, $\alpha_n = \tilde{\alpha}_n$, $\beta_n = \tilde{\beta}_n$, $\gamma_n = \tilde{\gamma}_n$ and $|\alpha_n| = F(n-1) - 2$, $|\beta_n| = F(n-2) + 2$, $|\gamma_n| = F(n) - 2$. If the words w_n begin always with a same letter ('b' in our case) then $w_n = f_n$ $(n \geq 5)$.

Proof. The proof is by induction on the integer n. Let us first show that the previous proposition holds for $n = 5$ and $n = 6$. From now on for simplicity we shall drop in the words $w_n, \alpha_n, \beta_n, \gamma_n, c_n$ the subscript n.

If $n = 5$, $F(5) = 5$, $|\alpha| = 1$, $|\beta| = 4$, $|\gamma| = 3$ and $|c| = 2$. Thus $\alpha = b$. The equation $w = b$, $\beta = \gamma c$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ has the only solution:

$\gamma = bab, \qquad \beta = abba, \qquad c = ba,$

if one wants that w contains two letters at least. Hence $w = f_5$.

If $n = 6$, $F(6) = 8$, $|\alpha| = 3$, $|\beta| = 5$, $|\gamma| = 6$ and $|c| = 2$. In this case one easily verifies that the only solution of the equation: $w = \alpha\beta = \gamma c$, where $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$, $\text{alph}(w) \geq 2$ is given by:

$\alpha = bab, \qquad \beta = babab, \qquad \gamma = (bab)^2, \qquad c = ab,$

so that $w = f_6$.

Let us now suppose $n \geq 7$ and consider the equation:

$$w = \alpha\beta = \gamma c, \qquad \alpha = \tilde{\alpha}, \qquad \beta = \tilde{\beta}, \qquad \gamma = \tilde{\gamma}, \tag{1}$$

with $|c| = 2$, $|\alpha| = F(n-1) - 2$, $|\beta| = F(n-2) + 2$ and $\text{alph}(w) \geq 2$. Let us write w as $w = w'w''$, with $|w'| = F(n-1)$ and $|w''| = F(n-2)$. We shall prove that $w' = f_{n-1}$ and $w'' = f_{n-2}$.

Since $n \geq 7$ it follows $|\beta| \geq 7$ so that being $|c| = 2$ one has: $\beta = \tilde{c}\delta c$ with $\tilde{\delta} = \delta$ and $|\delta| = F(n-2) - 2 \geq 3$. Thus from (1) we can write:

$$w = \alpha\tilde{c}\delta c = \gamma c, \tag{2}$$

and

$w' = \alpha c, \qquad w'' = \delta c.$

Hence w' has a palindrome left-factor α whose length $|\alpha| = |w'| - 2$. Moreover from (2) one has:

$$\gamma = \alpha\tilde{c}\delta = \delta c\alpha. \tag{3}$$

Since $F(n) > 2F(n-2)$ it follows that $|\gamma| > 2|\delta|$ so that from (3) one has $\gamma = \delta\epsilon\delta$ with $\epsilon = \tilde{\epsilon}$ and then

$w' = \alpha\tilde{c} = \delta\epsilon$.

The word w' is then the product of the two palindrome words δ and ϵ of lengths $|\delta| = F(n-2) - 2$, $|\epsilon| = |w'| - |\delta| = F(n-3) + 2$. Moreover $\text{alph}(w') \geq 2$. In fact, otherwise, $w' = b^{|w'|}$, $\gamma = b^{|\gamma|}$, $c = b^2$ and $w = \gamma c = b^{|w|}$ which is a contradiction. By the hypothesis of the induction it follows that $w' = f_{n-1}$.

Let us now prove that $w'' = f_{n-2}$. The word $w'' = \delta c$ has the palindrome left-factor δ of length $|\delta| = F(n-2) - 2$. Moreover the first letter of w'' is the first letter of δ that is 'b' (cf. (3)). We can rewrite (3) as:

$\alpha(\tilde{w}'') = w''\alpha$.

From the solution of the equation $xy = yz$, $x, y, z \in A^*$, in free monoids [5] one easily derives that (cf. [2]):

$$w'' = \delta c = \zeta\theta, \qquad (\tilde{w}'') = \theta\zeta,$$

$$\alpha = (\zeta\theta)^k\zeta, \quad k \geq 0,$$

$$\zeta = \tilde{\zeta}, \qquad \theta = \tilde{\theta} \neq 1.$$

Moreover one has that:

$$|\zeta| + |\theta| = |\delta| + 2 = F(n-2),$$

$$|\alpha| = (k+1)|\zeta| + k|\theta| = F(n-1) - 2.$$

It follows that:

$$|\zeta| = (1-k)\,F(n-2) + F(n-3) - 2,$$

$$|\theta| = kF(n-2) - F(n-3) + 2.$$

If $k = 0$ one has that $|\theta| = -F(n-3) + 2 \leq -F(4) + 2 = -1$ which is absurd. If $k \geq 2$ then $|\zeta| < 0$ which is also absurd. Thus the only remaining possibility is $k = 1$ so that:

$$|\zeta| = F(n-3) - 2, \qquad |\theta| = F(n-4) + 2. \qquad (4)$$

Thus w'' is the product of the two palindrome words ζ and θ of lengths given by (4). Finally $\text{alph}(w'') \geq 2$. In fact if $\text{alph}(w'') = 1$ then $w'' = \delta c = b^{|w''|}$. This would imply $c = b^2$, $\beta = \tilde{c}\delta c = b^{|\beta|}$. From (2), $w = \alpha b^{|\beta|} = \gamma b^2$ so that $\alpha b^{|\beta|-2} = \gamma = b^{|\beta|-2}\alpha$. Hence $\alpha = b^{|\alpha|}$, $\gamma = b^{|\gamma|}$ and $w = b^{|w|}$ which is a contradiction. By making use of the hypothesis of the induction it follows that $w'' = f_{n-2}$. Thus $w = w'w'' = f_{n-1}f_{n-2} = f_n$.

References

[1] J. Berstel, Private communication.

[2] A. De Luca, On some combinatorial problems in free monoids, Discrete Math., in press.

[3] J.P. Duval, Contribution à la combinatoire du monoide libre, Thèse d'Etat, Université de Rouen (1980).

[4] D.E. Knuth, J.H. Morris and V.R. Pratt, Fast pattern matching in strings, SIAM J. Comput. 6 (1977) 323–350.

[5] A. Lentin, Equations dans les Monoides Libres (Gauthier-Villars, Paris, 1972).

Aldo with flu during a MFCS Congress
in a Student Accomodation in Eastern Europe.

Since then, when we met, he always said:
"Grazie, per avermi salvato la vita!"
(Thanks for saving my life!)

The n-th letter in a Fibonacci word

n	w_n	length(w_n)	
1	a	1	1st = 'a'
2	b	1	2-1 = 1
3 = 2 1	b a	2	2nd
4 = 3 2	ba b	3	2nd
5 = 4 3	bab ba	5	2nd
6 = 5 4	babba bab	8	2nd
7 = 6 5	babbabab babba	13	15-13 = 2
8 ...		21	15th
9 ...		34	15th
10 ...		55	15th
...			

The 15^{th} letter of w_{10} is 'a'.

The k^{th} letter of w_n is computed in O(n) time (using sums and subtractions only).