Theory of Markoff
and
Christoffel words

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1. Markoff equation

Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz$$

A Markoff triple is a multiset

$$\{x, y, z\}$$

satisfying this equation.

Examples

$$\left.\begin{array}{l}\{1, 1, 1\} \\ \{1, 1, 2\}\end{array}\right\} \text{improper triples (or singular)}$$

$$\{1, 2, 5\}$$

$$\{1, 13, 5\}$$

$$\{2, 5, 29\}$$

$$\{5, 13, 194\}$$

A <u>Markoff number</u> is an element of a Markoff triple

Sequence of Markoff numbers:

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, ...

(includes Fibonacci and Pell numbers of even rank)

2. Frobenius conjecture (1913)

It is known that for each Markoff number m, there exists a Markoff triple whose maximum is m.

The conjecture, also called Markoff numbers injectivity conjecture is:

is this triple unique?

There are many equivalent formulations (one below).

Many partial results, see the book by Martin Aigner, Markov's theorem and 100 years of the uniqueness conjecture.

3. Markoff theory for approximations

Hurwitz $\sqrt{5}$ theorem:

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$ There exist infinitely many rationals $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\, q^2}$$

(extends Dirichlet: infinitely many $\frac{p}{q}$ such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$)

cf. convergents of continued fraction for α

Hurwitz $\sqrt{5}$ theorem is the first step in the theory of Markoff for approximations

Recall that two numbers

$$\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$$

have ultimately the same expansion into continued fractions

iff

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$, one has

$$\beta = \frac{a\alpha + b}{c\alpha + d}$$

(Serret's theorem)

<u>Def.</u> α, β are called <u>equivalent</u>

<u>Example</u> $\alpha = \frac{\sqrt{5}+1}{2} = [1, 1, 1, \ldots] = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$

$$\beta = [a_0, \ldots, a_n, 1, 1, 1, \cdots]$$

Second step of Markoff theory

If α is not equivalent to the golden ratio $\frac{\sqrt{5}+1}{2}$, then there are infinitely $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{8}\, q^2}$$

(better approximation than $< \frac{1}{\sqrt{5}\, q^2}$)

Third step of Markoff theory

If α is not equivalent to $\frac{\sqrt{5}+1}{2}$, nor to $\sqrt{2}+1 = [2,2,2,\ldots] = 2 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{\ddots}}}$,

then there are infinitely many $\frac{p}{q}$ such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\frac{\sqrt{221}}{5}\, q^2}$$

Markoff theory continues so on:

there are infinitely many (GL$_2$-classes of) exceptions and better approximations.

The constants C at the denominator of

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{C q^2}$$

are larger and larger:

$$\sqrt{5} < \sqrt{8} < \frac{\sqrt{221}}{5} < \frac{\sqrt{1517}}{13} < \cdots\cdots < 3$$

They are parametrized by the Markoff numbers m:

$$C = \sqrt{9 - \frac{4}{m^2}}$$

(ex. $m = 1 \quad C = \sqrt{5}$

$m = 2 \quad C = \sqrt{8}$

$m = 5 \quad C = \sqrt{9 - \frac{4}{25}} = \frac{\sqrt{221}}{5}$)

The exceptions are also linked with Markoff numbers. Precisely, they are determined by Christoffel words (to be defined below) as follows:

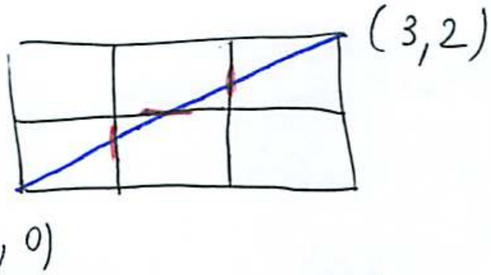
$$\chi : \{a, b\}^* \longrightarrow \{1, 2\}^*$$

$$a \longmapsto 11$$

$$b \longmapsto 22$$

Christoffel word			Exception
a	$\xrightarrow{\chi}$	11	$[1,1,1,\dots] = [\overline{1}]$
b	$\longrightarrow$	22	$[2,2,\dots] = [\overline{2}]$
ab	$\longrightarrow$	1122	$[\overline{1,1,2,2}]$
aab	$\longrightarrow$	111122	$[\overline{1,1,1,1,2,2}]$
abb	$\longrightarrow$	112222	$[\overline{1,1,2,2,2,2}]$
$\vdots$			

4. Words: central, Christoffel, standard

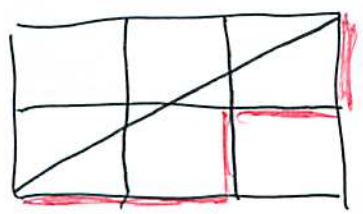


central word = intersection word (billiard word)   aba

To each (p, q), $p \perp q$, associate a central word $u \in \{a, b\}^*$

It is a palindrome.

The word aub is called a Christoffel word (1875).



← discretizing of the diagonal

$a\,a\,b\,a\,b$

$= a\,(aba)\,b$

u central word

uab, uba are called

<u>standard words</u> by

Aldo de Luca.

They arise from his study of the Fibonacci word (Infor. Proc. Letters 1981) and a conjecture of Robinson* (1986), solved by Pedersen (1988).

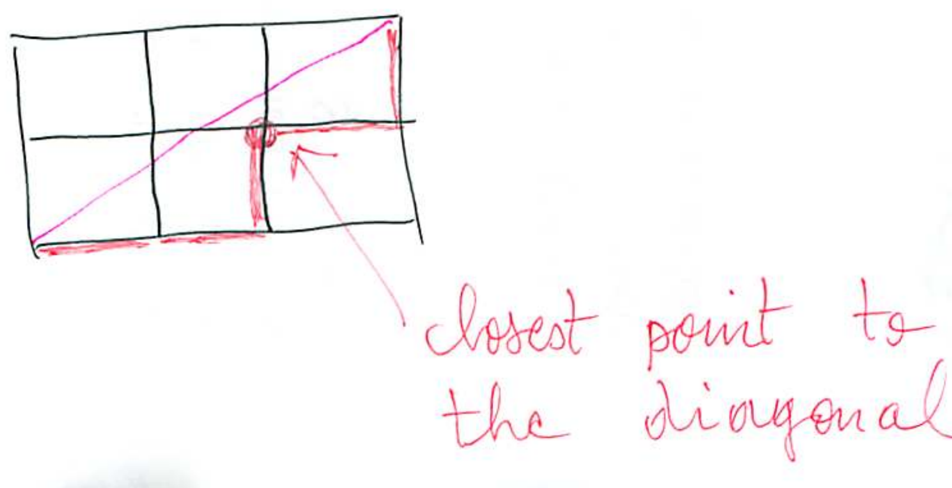
Mignosi, de Luca (1994): a word w is standard iff $w = pq = qab$ where p, q, r are palindromes.

* inspired by Aldo de Luca, see de Luca 1994

Variant of this result:
Each Christoffel word is uniquely the product of two Christoffel words (Borel, Laubie 1993): this is called the <u>standard factorization</u>

Ex $\quad aabab = aab \cdot ab$

$aabaabab = aab \cdot aabab$



5. Parametrization of Markoff triples

$$\mu : \{a, b\}^* \longrightarrow SL_2(\mathbb{N})$$

$$\mu a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mu b = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Theorem (Cohn 1955, Bombieri 2007, R. 2009)

Define a mapping from the set of Christoffel words into the set of triples of numbers:

$$w \longmapsto \left\{ \frac{1}{3}\, tr(\mu w),\ \frac{1}{3}\, tr(\mu u),\ \frac{1}{3}\, tr(\mu v) \right\}$$

where $w = uv$, standard factorization.

This mapping is a bijection from the set of Christoffel words onto the set of proper Markoff triples.

Addendum: $tr(\mu w) = 3 (\mu w)_{12}$

Examples

- $ab = a \cdot b$

$$\mu(ab) = \begin{pmatrix} 15 & 5 \\ 7 & 3 \end{pmatrix} \quad \mu a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \mu b = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

$\{5, 1, 2\}$ Markoff triple

- $aabab = aab \cdot ab$

$$\mu(aabab) = \begin{pmatrix} 463 & 194 \\ 284 & 119 \end{pmatrix} \quad \mu(aab) = \begin{pmatrix} 31 & 13 \\ 19 & 8 \end{pmatrix}$$

$$\mu(ab) = \begin{pmatrix} 15 & 5 \\ 7 & 3 \end{pmatrix}$$

$\{194, 13, 5\}$ Markoff triple

<u>Corollary</u> $w \mapsto (\mu w)_{12} = \frac{1}{3}\,\mathrm{tr}(\mu w)$

is a surjective mapping from Christoffel words to Markoff numbers.

Frobenius conjecture

$\Longleftrightarrow$

injective mapping

One idea of the proof of the theorem (due to Cohn):

Fricke relation (1896)

$A, B \in SL_2$

$$\mathrm{Tr}(A)^2 + \mathrm{Tr}(B)^2 + \mathrm{Tr}(AB)^2$$

$$= \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(AB) + \mathrm{Tr}(ABA^{-1}B^{-1}) + 2$$

6. Christoffel pairs

$w = uv$

w, u, v Christoffel words

(u, v) is called a <u>Christoffel pair</u>

Ex (aab, ab)

<u>Th</u>. Christoffel pairs are bases of the free group $F(a, b)$

(Borel, Laubie 1994)

Equivalent to a result of
de Luca, Mignosi 1994
Berstel, de Luca 1997

on standard pairs.

7. Commutator subgroup $SL_2(\mathbb{Z})'$

It is known that the subgroup of $SL_2(\mathbb{Z})$ generated by the matrices $ABA^{-1}B^{-1}$, $A, B \in SL_2(\mathbb{Z})$, is a free group on two generators.

Actually $\mu : \overset{F(a,\, b)}{\cancel{\{a, b\}^*}} \longrightarrow SL_2(\mathbb{Z})'$

is an isomorphism

$$\mu a = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mu b = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

If (u, v) is a Christoffel pair, it is a basis of $F(a, b)$, hence $(\mu u, \mu v)$ is a basis of the free group $SL_2(\mathbb{Z})'$.

8. Perrine's theorem (2002)

Theorem

For each basis (A, B) of the free group $SL_2(\mathbb{Z})'$, the triple

$$\left\{ \frac{1}{3} \operatorname{tr}(A), \frac{1}{3} \operatorname{tr}(B), \frac{1}{3} \operatorname{tr}(AB) \right\}$$

is a Markoff triple.

(we already know that each Markoff triple is attained)

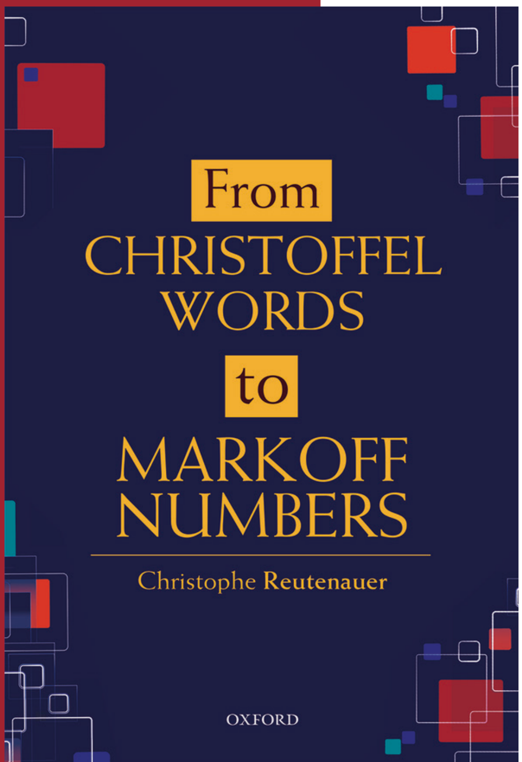
Ingredient of the proof: Fricke relation

and

Nielsen's criterion (1918):
(u, v) pair of elements of $F(a, b)$ is a basis of $F(a, b)$
iff
$uvu^{-1}v^{-1}$ is conjugate to $(aba^{-1}b^{-1})^{\pm 1}$

Grazie !

Book | C. Reutenauer
From Christoffel words to Markoff numbers
Oxford University Press
2019



From Christoffel Words to Markoff Numbers

Christophe Reutenauer, Université du Québec à Montréal

- Introductory textbook for students with a general basis in mathematics
- For the first time in literature on the subject, this textbook treats the two aspects of Markoff's Theory simultaneously
- Numerous figures throughout the book help to illustrate its points and provide proofs of discrete geometry

The link between Christoffel words and the Markoff theory was noted by Ferdinand Frobenius in 1913, but has been neglected in recent times. Motivated by this overlooked connection, this book looks to expand on the relationship between these two areas. Part 1 focuses on the classical theory of Markoff, while Part II explores the more advanced and recent results of the theory of Christoffel words.

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