Interconnections between the Structure Theory on Set Addition and Rewritability in Groups

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Titolo

Connessioni tra

la teoria strutturale della

Somma tra insiemi

e la Riscrivibilità nei gruppi

Dedication

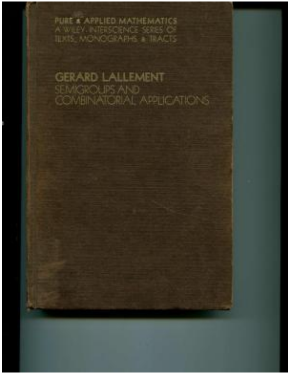
Dedicated to the memory of **Aldo de Luca**

Some memories

Some memories

Some memories

Some memories



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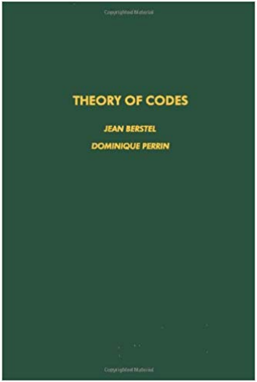

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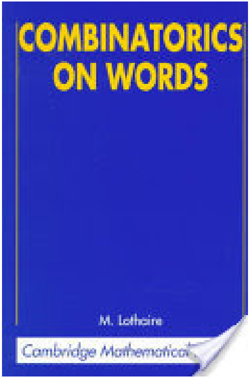

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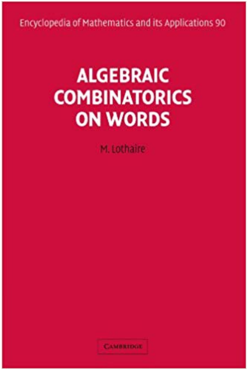
Addison-Wesley, Reading, 1983



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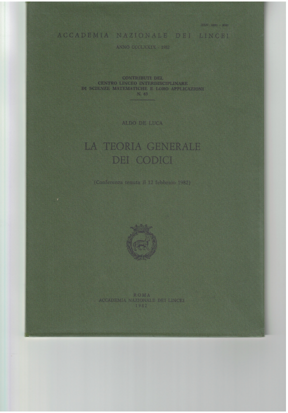
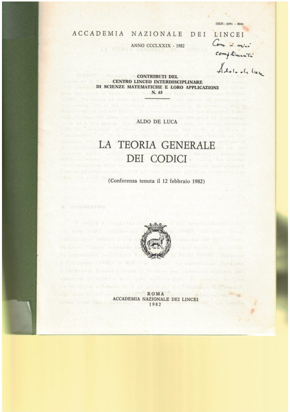
Some memories





Some memories

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Teoria degli Automi Finiti
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Some memories

Some memories

Key definitions - totally n-rewritability

Let $(S, \cdot)$ be a semigroup, $n \geq 2$ an integer.

Definition

S is said to be **totally n-rewritable** ($S \in \mathcal{P}_n$) if, for each n-ple $(x_1, x_2, \ldots, x_n)$ of elements of S, there is a non-trivial permutation σ of the set $\{1, 2, \ldots, n\}$ such that

$$x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

Key definitions - n-rewritability

Let $(S, \cdot)$ be a semigroup, $n \geq 2$ an integer.

Definition

S is said to be **n-rewritable** ($S \in \mathcal{Q}_n$)
if, for any elements $x_1, x_2, \dots, x_n$ in S,
there exist permutations σ and τ
of $\{1, 2, \dots, n\}$, $\sigma \neq \tau$, such that

$$x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)} = x_{\tau(1)} x_{\tau(2)} \cdots x_{\tau(n)}.$$

Some more properties - $\mathcal{P}_n$-sequenceability

Let $(G, \cdot)$ be a group, $n \geq 2$ an integer.

Definition

G is said to be **$\mathcal{P}_n$-sequenceable** if we can write the elements of G in a sequence $(x_\alpha)_{\alpha \in \Lambda}$, Λ a well-ordered set, such that for every $\alpha \in \Lambda$, $\alpha + n - 1 \leq max\Lambda$ if this exists, the product $x_\alpha x_{\alpha+1} \cdots x_{\alpha+n-1}$ can be rewritten in at least one way, *i.e.* there exists a non-trivial permutation σ of the set $\{0, 1, \ldots, n-1\}$ such that

$$x_\alpha x_{\alpha+1} \cdots x_{\alpha+n-1} = x_{\alpha+\sigma(0)} x_{\alpha+\sigma(1)} \cdots x_{\alpha+\sigma(n-1)}.$$

Some more properties - $\mathcal{P}_n$-sequenceability

Theorem

Every group G is $\mathcal{P}_4$-**sequenceable**.

Theorem

Every countably infinite group G is $\mathcal{P}_3$-**sequenceable**.

P. L., M. Maj, Some remarks on $\mathcal{P}_n$-sequenceable groups, *Arch. Math.* **60** (1993), 15-19.

Some more properties - $\mathcal{R}_n$

Let $(G, \cdot)$ be a group, $n \geq 2$ an integer.

Definition

G is said to be an $\mathcal{R}_n$-**group**
if every infinite subset X of G contains
a subset $\{x_1, \ldots, x_n\}$ of n elements such that

$$x_1 x_2 \cdots x_n = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

for some non-trivial permutation σ.

Some more properties - $\mathcal{R}_n$

Theorem

A group G is an $\mathcal{R}_n$-group for some integer n
if and only if G has a normal subgroup F such that
G/F is finite, F is an FC-group and
the exponent of $F/Z(F)$ is finite.

M. Curzio, P. L., M. Maj, A. Rhemtulla, GROUPS WITH MANY REWRITABLE PRODUCTS, *Proc. Amer. Math. Soc.* **115** no. 4 (1992), 931-934.

Some more properties - Commutators

Definition

Let G be a group and let $x_1, x_2, \ldots$ be elements of G.

The commutator of x_1 and x_2 is

$$[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2.$$

More generally, a simple commutator of weight $n \geq 2$ is defined recursively by the rule

$$[x_1, \ldots, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

Some more properties - $\mathcal{C}_n$

Definition

A group G is said to be a $\mathcal{C}_n$-**group**

if, for each n-ple $(x_1, x_2, \ldots, x_n)$ of elements of G,

there is a non-trivial permutation σ

of the set $\{1, 2, \cdots, n\}$ such that

$$[x_1, \ldots, x_n] = \left[x_{\sigma(1)}, \ldots, x_{\sigma(n)}\right].$$

Results in

P. L., ON GROUPS WITH A PERMUTATIONAL PROPERTY ON COMMUTATORS, *Proc. "GROUPS - KOREA 1988"*, Lecture Notes in Mathematics - Springer **1398** no. 4 (1989), 110-116.

Some more properties - $\mathcal{PSP}_n$

Definition

Let G be a group and $n \geq 2$.

The group G is said to be a **$\mathcal{PSP}_n$-group**

if, for each n-ple $(H_1, H_2, \ldots, H_n)$ of subgroups of G,

there is a non-trivial permutation σ

of the set $\{1, 2, \ldots, n\}$ such that

$$H_1 \cdots H_n = H_{\sigma(1)} \cdots H_{\sigma(n)}.$$

Some more properties - $\mathcal{PSP}_n$

Results in

A.H. Rhemtulla, A.R. Weiss, Groups with permutable subgroup products, *Proc. 1987 Singapore Conference in Group Theory*, Walter der Gruyter, Berlin, New York (1989), 485-495.

M. Maj, Some remarks on groups with permutable subgroup products, *Comm. Algebra*, **17** no. 10 (1989), 2539-2555.

P. L., M. Maj, A. Rhemtulla, Periodic groups with permutable subgroup products, *Math. Proc. Camb. Phil. Soc.*, **106** (1989), 431-437.

P. L., M. Maj, A. Rhemtulla, Residually Solvable $\mathcal{PSP}$-Groups, *Boll. U.M.I.*, **7** no. 7-B (1993), 253-261.

Starting point - Freiman-Schein's paper

Gregory A. Freiman, Boris M. Schein

INTERCONNECTIONS BETWEEN THE STRUCTURE THEORY OF SET ADDITION AND REWRITABILITY IN GROUPS

Proc. Amer. Math. Soc. **113** no. 4 (1991), 899-910.

Starting point - Freiman-Schein's paper

Gregory A. Freiman, Boris M. Schein
Interconnections between the Structure Theory on Set Addition and Rewritability in Groups
Proc. Amer. Math. Soc. **113** no. 4 (1991), 899-910.

Abstract

An approach to groups and semigroups stemming from the structure theory of set addition turns out to have much in common with the so-called permutation or rewritable properties. We explain these connections and show how these properties take their place in a wider class of interesting and naturally arising problems.

. . .

Introduction

In recent years semigroups and groups satisfying the so-called permutation or rewritable properties attracted considerable attention.

. . .

Problems connected with permutation and rewritable properties of groups and semigroups find their natural place in the structure theory of set addition.

The goal of this paper is to show how the problems of rewritability and a class of analogous problems can be approached from the point of view of the structure theory of set addition. This approach gives rise to many new and natural questions about rewritability and may indicate possible ways of solving these problems.

Starting point - Freiman-Schein's paper

We are grateful to Professors **Ya. G. Berkovich** and **D. Gorenstein** for drawing our attention to possible connections between the structure theory of set addition and rewritability.

Additive Number Theory

Set Addition

Gregory A. Freiman,
Foundations of a structural theory of set addition
Kazan, 1966 (Russian); English transl.: Translations of mathematical monographs, **37**, American Mathematical Society, Providence, Rhode Island, 1973.

Additive Number Theory

Set Addition

M.B. Nathanson
Additive Number Theory - Inverse Problems and the Geometry of Sumsets
Springer, New York, 1996.

A. Geroldinger, I.Z. Ruzsa,
Combinatorial Number Theory and Additive Group Theory
Birkäuser, Basel - Boston - Berlin, 2009.

Basic definition

Definition

If $X_1, \dots, X_n$ are sets of integers $(n \geq 2)$, then we put

$$X_1 + \cdots + X_n := \{x_1 + \cdots + x_n \mid x_1 \in X_1, \dots, x_n \in X_n\}.$$

If X is a set of integers and $X_1 = \cdots = X_n = X$,

then we denote the set $X_1 + \cdots + X_n$ by nX.

The set $X_1 + \cdots + X_n$ is also called

the (Minkowski) sumset of $X_1, \dots, X_n$.

Basic definition

Main Problem

Let X *be a finite set of integers and consider*

$$2X := \{x_1 + x_2 \mid x_1, x_2 \in X\} .$$

What can be said about $2X$ *if we know some property of* X *?*

What can be said about X *if we have some bound for* $|2X|$ *?*

Background

Remark (1)

Let X be a finite set of integers with k elements. Then

$$|2X| \geq 2k - 1.$$

Proof. Let $X = \{x_1, x_2, \ldots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$.

Clearly

$$2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$$

and each of these elements belongs to $2X$. Hence $|2X| \geq 2k - 1$, as required. //

Background

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and each of these elements belongs to $2X$. Hence $|2X| \geq 2k - 1$, as required. //

Background

Remark (2)

Let X *be a finite set of integers with* k *elements.*
If X *is an* ***arithmetic progression***:

$$X = \{a, a + r, a + 2r, \ldots, a + (k-1)r\},$$

then

$$|2X| = 2k - 1.$$

Proof. We have

$$2X = \{2a, 2a + r, 2a + 2r, ..., 2a + (2k-2)r\}.$$

//

Background

Remark (2)

Let X be a finite set of integers with k elements.
If X is an ***arithmetic progression****:*

$$X = \{a, a+r, a+2r, \dots, a+(k-1)r\},$$

then

$$|2X| = 2k-1.$$

Proof. We have

$$2X = \{2a, 2a+r, 2a+2r, ..., 2a+(2k-2)r\}.$$

//

Remark (3)

Let X be a finite set of integers with k elements.
If $|2X| = 2k - 1$, then X is an arithmetic progression.

Proof. Let $X = \{x_1, x_2, \dots, x_k\}$, and assume $x_1 < x_2 < \dots < x_k$. Then $2X = \{2x_1, x_1 + x_2, 2x_2, x_2 + x_3, 2x_3, \dots, 2x_{k-1}, x_{k-1} + x_k, 2x_k\}$ with $2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \dots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$.
Clearly $x_2 = x_1 + (x_2 - x_1)$.
It holds $2x_1 < x_1 + x_3 < 2x_3$ with $x_1 + x_3 \neq x_1 + x_2, x_2 + x_3$. Therefore $x_1 + x_3 = 2x_2$ and $x_3 = 2x_2 - x_1 = x_2 + (x_2 - x_1)$. Analogously $x_2 + x_4 = 2x_3$ and $x_4 = 2x_3 - x_2 = x_3 + (x_3 - x_2) = x_3 + (x_2 - x_1)$, and so on. //

Remark (3)

Let X be a finite set of integers with k elements.
If $|2X| = 2k - 1$, then X is an arithmetic progression.

Proof. Let $X = \{x_1, x_2, \ldots, x_k\}$, and assume $x_1 < x_2 < \cdots < x_k$. Then $2X = \{2x_1, x_1 + x_2, 2x_2, x_2 + x_3, 2x_3, \ldots, 2x_{k-1}, x_{k-1} + x_k, 2x_k\}$ with
$2x_1 < x_1 + x_2 < 2x_2 < x_2 + x_3 < 2x_3 < \cdots < 2x_{k-1} < x_{k-1} + x_k < 2x_k$.
Clearly $x_2 = x_1 + (x_2 - x_1)$.
It holds $2x_1 < x_1 + x_3 < 2x_3$ with $x_1 + x_3 \neq x_1 + x_2, x_2 + x_3$. Therefore
$x_1 + x_3 = 2x_2$ and $x_3 = 2x_2 - x_1 = x_2 + (x_2 - x_1)$. Analogously
$x_2 + x_4 = 2x_3$ and $x_4 = 2x_3 - x_2 = x_3 + (x_3 - x_2) = x_3 + (x_2 - x_1)$,
and so on. //

Background

Gregory A. Freiman, *Structure theory of set addition*, Astérisque, **258** (1999), 1-33.

"Thus a **direct problem** *in additive number theory is a problem which, given summands and some conditions, we discover something about the set of sums. An* **inverse problem** *in additive number theory is a problem in which, using some knowledge of the set of sums, we learn something about the set of summands."*

Basic definition

Definition

If X is a subset of a group $(G, +)$, write

$$2X = X + X := \{x + y \mid x, y \in X\}.$$

$2X$ is also called the double of X.

If G is a multiplicative group, then we put

$$X^2 = XX := \{xy \mid x, y \in X\}.$$

X^2 is also called the square of X.

Background - some direct results

Remark

Let X be a finite subset of a group G. Then

$$|X| \leq |X^2| \leq |X|^2.$$

The bounds are sharp.

Background - some direct and inverse results

Example

If X is a subgroup of G, then $X^2 = X$.

More generally, if $X = xH$, where H is a subgroup of G and $xH = Hx$, then $X^2 = xHxH = x^2H$ and $|X^2| = |X|$.

Proposition

Let X be a non-empty finite subset of a group G.

$|X^2| = |X|$ if and only if $X = xH$, where $H \leq G$ and $xH = Hx$.

Results - Freiman-Schein's paper

Let $(S, \cdot)$ be a semigroup, $n \geq 2$ an integer.

Definition

Let X a subset of S.

Write

$$X^{[n]} = \{x_1 \cdots x_n | x_1, \ldots, x_n \in X, x_i \neq x_j \text{ for } i \neq j\}.$$

Remark

S is n-rewritable if and only if

$|X^{[n]}| < n!$ for every subset X of S of order n.

Results - Freiman-Schein's paper

Let $(S, \cdot)$ be a semigroup, $n \geq 2$ an integer.

Definition

Let X a subset of S.

Write

$$X^{[n]} = \{x_1 \cdots x_n | x_1, \ldots, x_n \in X, x_i \neq x_j \textit{ for } i \neq j\}.$$

Remark

S is n-rewritable if and only if

$|X^{[n]}| < n!$ for every subset X of S of order n.

Results - Freiman-Schein's paper

Let $(G, \cdot)$ be a group, $1 \leq n \leq 6$ an integer.

Definition

G is said to be an $\mathcal{R}(3, n)-$group (or $G \in \mathcal{R}(3, n)$)

if $|X^{[3]}| \leq n$ for every subset X of G of order 3.

Theorem

G is 3-totally rewritable if and only if

G is an $\mathcal{R}(3, 2)-$group.

Results - Freiman-Schein's paper

Let $(G, \cdot)$ be a group, $1 \leq n \leq 6$ an integer.

Definition

G is said to be an $\mathcal{R}(3, n)-$group (or $G \in \mathcal{R}(3, n)$)
if $|X^{[3]}| \leq n$ for every subset X of G of order 3.

Theorem

G is 3-totally rewritable if and only if
G is an $\mathcal{R}(3, 2)-$group.

Connections - the small squaring property

Let $(G, \cdot)$ be a group, $m \geq 2$ an integer.

Definition

G is said to have the square property on $m-$sets

(or $G \in \mathcal{DS}(m)$)

if $|X^2| < m^2$ for every subset X of G of order m.

Theorem (G.A. Freiman, 1981)

$G \in \mathcal{DS}(2)$ if and only if G is a Dedekind group.

Connections - the small squaring property

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Theorem (G.A. Freiman, 1981)

$G \in \mathcal{DS}(2)$ if and only if G is a Dedekind group.

Connections - the small squaring property

Theorem

Groups in $\mathcal{DS}(3)$ have been classified.

J.G. Berkovich, G.A. Freiman, C.E. Praeger, Small squaring and cubing properties of finite groups, *Bull. Austral. Math. Soc.* **44** (1991), 429-450.

P. L., M. Maj, The classification of groups with the small squaring property on 3-sets, *Bull. Austral. Math. Soc.* **46** (1992), 263-269.

Connections - the small squaring property

Theorem

A group G is in $\mathcal{DS}(m)$ for some $m \geq 2$

if and only if

either G is nearly-dihedral or $G^{(2)}$ is of finite order.

M. Herzog, P. L., M. Maj, On a combinatorial problem in group theory, *Israel J. Math.* **82** (1993), 329-340.

Definitions

A group G is said to have the nearly-dihedral if it contains a normal abelian subgroup H of finite index, on which each element of G acts by conjugation either as the identity or as the inverting automorphism.

We denote by $G^{(2)}$ the subgroup of G generated by the squares of all elements of G.

Problems

Main Problem

What happens if G is any torsion-free group?

Which bounds can we get on $|X^2|$ if X is a finite subset of a torsion-free group?

What about inverse results for G torsion-free?

Background - direct results

Proposition

If X is a non-empty finite subset of the group of the integers, then we have

$$|2X| \geq 2|X| - 1.$$

More generally:

Theorem (J.H.B. Kemperman, Indag. Mat., 1956)

If X is a non-empty finite subset of a torsion-free group, then we have

$$|X^2| \geq 2|X| - 1.$$

A definition

Definition

If $a, r \neq 1$ are elements of a multiplicative group G, a geometric left (rigth) progression with ratio r and length n is the subset of G

$$\{a, ar, ar^2, \cdots, ar^{n-1}\} \ (\{a, ra, r^2a, \cdots, r^{n-1}a\}).$$

If G is an additive abelian group

$$\{a, a+r, a+2r, \cdots, a+(n-1)r\}$$

is called an arithmetic progression with difference r and length n.

An example

Example

If $X = \{a, ar, ar^2, \cdots, ar^{n-1}\}$ is a geometric progression in a torsion-free group and $ar = ra$, then

$$X^2 = \{a^2, a^2r, a^2r^2, \cdots, a^2r^{2n-2}\} \text{ has order } 2|X| - 1.$$

Background

Theorem (G.A. Freiman, B.M. Schein, Proc. Amer. Math. Soc., 1991)

If X is a finite subset of a torsion-free group, $|X| = k \geq 2$,

$$|X^2| = 2|X| - 1$$

if and only if

$X = \{a, aq, \cdots, aq^{k-1}\}$, *and either* $aq = qa$ *or* $aqa^{-1} = q^{-1}$.

In particular, if $|X^2| = 2|X| - 1$, then X is contained in a left coset of a cyclic subgroup of G.

Background - inverse results

Theorem (Y.O. Hamidoune, A.S. Lladó, O. Serra, Combinatorica, 1998)

If X is a finite subset of a torsion-free group, $|X| = k \geq 4$, then

$$|X^2| \leq 2|X|$$

if and only if *there exist $a, q \in G, aq = qa$ such that*

$$X = \{a, aq, \cdots, aq^k\} \setminus \{c\}, \text{ with } c \in \{a, aq\}.$$

Doubling problems

Let G be a group and X a finite subset of G.
Let α, β be real numbers.

Problem

What is the structure of X if $|X^2|$ satisfies

$$|X^2| \leq \alpha|X| + \beta \text{ ?}$$

Problems of this kind are called **inverse problems of doubling type** in additive number theory. The coefficient α, or more precisely the ratio $\frac{|X^2|}{|X|}$, is called the **doubling coefficient** of X.

Inverse problems of doubling type have been first investigated by **G.A. Freiman**.

Doubling property

There are two main types of questions one may ask.

Question

What is the general type of structure that X can have if

$$|X^2| \leq \alpha |X| + \beta?$$

How behaves this type of structure when α increases?

Studied recently by many authors:

E. Breuillard, B. Green, I.Z. Ruzsa, T. Tao, . . .

Very powerful general results have been obtained (leading to a qualitatively complete structure theorem thanks to the concepts of nilprogressions and approximate groups).

Small doubling problems

But these results are not very precise quantitatively.

Question

For a given (in general quite small) range of values for α find the precise (and possibly complete) description of those finite sets X which satisfy

$$|X^2| \leq \alpha|X| + \beta,$$

with α and $|\beta|$ **small**.

Problems of this kind are called **inverse problems of small doubling type**.

Small doubling problems

Theorem (G.A. Freiman, 1959)

Let X be a finite set of integers with $k \geq 3$ elements and suppose that

$$|2X| \leq 3k - 4.$$

Then X is contained in an arithmetic progression of size $2k - 3$:

$$\{a, a + q, a + 2q, \cdots, a + (2k - 4)q\}.$$

Small doubling problems

Conjecture (G.A. Freiman)

If G is any torsion-free group, X a finite subset of G, $|S| \geq 4$, and

$$|X^2| \leq 3|X| - 4,$$

then S is contained in a geometric progression of length at most $2|X| - 3$.

Small doubling problems

Theorem (G.A. Freiman)

Let X be a finite set of integers with $k \geq 2$ elements and suppose that

$$|2X| \leq 3k - 3.$$

Then one of the following holds:

(i) X is contained in an arithmetic progression of size at most $2k-1$;
(ii) X is a bi-arithmetic progression

$$X = \{a, a+q, a+2q, \cdots, a+(i-1)q\} \cup \{b, b+q, a+2q, \cdots, b+(j-1)q\};$$

(iii) $k = 6$ and X has a determined structure.

Small doubling problems

Problem

Let G be any torsion-free group, X a finite subset of G, $|X| \geq 3$.

What is the structure of X if

$$|X^2| \leq 3|X| - 3?$$

Small doubling problems

Freiman studied also the case $|2X| = 3|X| - 2$, X a subset of the integers. He proved that, with the exception of some cases with $|X|$ small, then either X is contained in an arithmetic progression or it is the union of two arithmetic progressions with same difference.

Conjecture (G.A. Freiman)

If G is any torsion-free group, X a finite subset of G, $|X| \geq 11$, and

$$|X^2| \leq 3|X| - 2,$$

then X is contained in a geometric progression of length at most $2|X| + 1$ or it is the union of two geometric progressions with same ratio.

Small doubling problems

By now, Freiman's theory had been extended tremendously, in many different directions.

It was shown by Freiman and others that problems in various fields may be looked at and treated as Structure Theory problems, including Additive and Combinatorial Number Theory, Group Theory, Integer Programming and Coding Theory.

J. Cilleuelo, **M. Silva**, **C. Vinuesa**, **H. Halberstam**, **N. Gill**, **B.J. Green**, **Helfgott**, **R. Jin**, **V.F. Lev**, **P. Y. Smeliansky** , **I.Z. Ruzsa**, **T. Sanders**, **T.C. Tao**, ...

Small doubling problems have been studied in abelian groups by many authors:
Y. O. Hamidoune, B. Green, M. Kneser, A.S. Lladó, A. Plagne, P.P. Palfy, I.Z. Ruzsa, O. Serra, Y.V. Stanchescu, . . .

Thank you for the attention !

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